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On the Location of Zeros of Analytic Functions

M. H. Gulzar
P. G. Department of Mathematics University of Kashmir, Srinagar 190006

Abstract: In this paper we consider a certain class of analytic functions whose coefficients are restricted to certain conditions, and find some interesting zero-free regions for them. Our results generalise a number of already known results in this direction.
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I. INTRODUCTION AND STATEMENT OF RESULTS

Regarding the zeros of analytic functions, Aziz and Shah [2]proved the following results:

Theorem A: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be a function analytic for $|z| \le t$ and for some $k \ge 1$,

$$ka_0 \ge ta_1 \ge t^2 a_2 \ge \ldots\ldots .$$

Then f(z) does not vanjsh in

$$\left| z-(\frac{k-1}{2k-1})t \right| \le \frac{kt}{2k-1} .$$

Theorem B: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be a function analytic for $|z| \le t$ and

$$\left|(a_1-ta_2)+(a_2-ta_3)z+(a_3-ta_4)z^2+\ldots..\right| \le \frac{M}{t} \text{ for } |z|=t .$$

Then f(z) does not vanish in $|z|<R$, where

$$R=\frac{1}{2M}\left\{-|a_0-ta_1|+\sqrt{|a_0-ta_1|^2-4|a_0|M}\right\} .$$

In this paper we are going to give generalizations of the above mentioned results. More precisely, we shall prove the following results:

Theorem 1: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be a function analytic for $|z| \le t$ and for some $\rho \ge 0$,

$$\rho + a_0 \ge ta_1 \ge t^2 a_2 \ge \ldots\ldots .$$

Then f(z) does not vanish in

$$\left| z-(\frac{\rho}{2\rho+a_0})t \right| < (\frac{\rho+a_0}{2\rho+a_0})t .$$

Remark 1: Taking $\rho=(k-1)a_0$, Theorem 1 reduces to Theorem A.

Taking $\rho=0$, we get the following result proved earlier by Aziz and Mohammad [1] :

Corollary 1: : Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be a function analytic for $|z| \le t$ and

$$0 < a_0 \ge ta_1 \ge t^2 a_2 \ge \ldots\ldots .$$

Then f(z) does not vanish in $|z|<t$.

Theorem 2: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be a function analytic for $|z| \leq t$ with

$a_j=\alpha_j e^{i\phi}+\beta_j e^{i\psi}$,j=0,1,2,……,n, and

$$\left|(\alpha_1-t\alpha_2)+(\alpha_2-t\alpha_3)z+(\alpha_3-t\alpha_4)z^2+\ldots..\right| \leq \frac{M_1}{t} \text{ for } |z|=t,$$

$$\left|(\beta_1-t\beta_2)+(\beta_2-t\beta_3)z+(\beta_3-t\beta_4)z^2+\ldots..\right| \leq \frac{M_2}{t} \text{ for } |z|=t.$$

Then f(z) does not vanish in $|z|<R$, where

$$R=\frac{1}{2(M_1+M_2)}\left\{-|a_0-ta_1|+\sqrt{|a_0-ta_1|^2+4|a_0|(M_1+M_2)}\ \right\}t\ .$$

Remark 2: If $\beta_j=0, \forall j=0,1,\ldots., n$, so that $M_2=0$, Theorem 2 reduces to Theorem B by taking $M_1=M$.

The following results are immediate consequences of Theorem 2:

Corollary 2: : Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic for $|z| \leq t$ with

$a_j=\alpha_j e^{i\phi}+\beta_j e^{i\psi}$,j=0,1,2,……,n, and for some $k \geq 1$

$$t\alpha_1 \leq t^2\alpha_2 \leq \ldots\ldots \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \ldots\ldots\ .$$

Then f(z) does not vanish in

$$|z|<\frac{1}{2M_k}\left\{-|a_0-ta_1|+\sqrt{|a_0-ta_1|^2+4|a_0|M_k)}\ \right\}t$$

where

$$M_k=2t^k\alpha_k-t\alpha_1+t|\beta_1|+2\sum_{j=2}^{\infty}|\beta_j|t^j\ .$$

Corollary 3: : Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic for $|z| \leq t$ with

$a_j=\alpha_j e^{i\phi}+\beta_j e^{i\psi}$,j=0,1,2,……,n, and for some $k \geq 1$

$$t\alpha_1 \leq t^2\alpha_2 \leq \ldots\ldots \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \ldots\ldots\ ,$$

$$t\beta_1 \leq t^2\beta_2 \leq \ldots\ldots \leq t^k\beta_k \geq t^{k+1}\beta_{k+1} \geq \ldots\ldots\ .$$

Then f(z) does not vanish in

$$|z|<\frac{1}{2M_k}\left\{-|a_0-ta_1|+\sqrt{|a_0-ta_1|^2+4|a_0|M_k)}\ \right\}t$$

where

$$M_k=2t^k\alpha_k-t\alpha_1+2t^k\beta_k-t\beta_1.$$

Taking k=1in Cor.2, and noting that $M_1=ta_1$ and $M_2=ta_2$,we get the following result from Cor.2:

Corollary 4 : Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic for $|z| \leq t$ with

$a_j=\alpha_j e^{i\phi}+\beta_j e^{i\psi}$,j=0,1,2,……,n, and

$$0 < t\alpha_1 \geq t^2\alpha_2 \geq ......,$$
$$0 < t\beta_1 \geq t^2\beta_2 \geq ....$$

Then f(z) does not vanish in

$$|z| < \frac{1}{2(\alpha_1+\beta_1)}\left\{-|a_0 - ta_1| + \sqrt{|a_0 - ta_1|^2 - 4|a_0|(\alpha_1+\beta_1)t)}\right\}.$$

Taking k=1and $\beta_j = 0, \forall j = 0,1,...., n$,in Cor.2, and noting that $M_1 = ta_1$, Cor.3 reduces to Cor.1.

II. PROOFS OF THEOREMS

Proof of Theorem 1: Since $f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0$ is analytic for $|z| \leq t$, therefore

$\lim_{j\to\infty} a_j t^j = 0$.Consider the function

$$\begin{aligned}
F(z) &= (z-t)f(tz) \\
&= (z-t)(a_0 + a_1tz + a_2t^2z^2 + ...) \\
&= -ta_0 + (a_0 - a_1t)z + (a_1t - a_2t^2)z^2 + ... \\
&= -ta_0 - \rho z + (\rho + a_0 - a_1t)z + (a_1t - a_2t^2)z^2 + ... \\
&= -ta_0 - \rho z + G(z),
\end{aligned}$$

where

$$G(z) = (\rho + a_0 - a_1t)z + z\sum_{j=2}^{\infty}(a_{j-1} - a_jt)z^{j-1}.$$

For $|z| = t$,

$$\begin{aligned}
|G(z)| &\leq (\rho + a_0 - a_1t)t + t[(a_1 - a_2t)t + (a_2 - a_3t)t^2 + ...] \\
&= (\rho + a_0)t.
\end{aligned}$$

Since f(z) is analytic for $|z| \leq t$, G(z) is analytic for $|z| \leq t$ and G(0)=0, we apply Schwarz lemma to G(z) to get

$$|G(z)| \leq (\rho + a_0)|z| \quad \text{for } |z| \leq t.$$

Hence it follows that

$$\begin{aligned}
|F(z)| &\geq |ta_0 + \rho z| - |G(z)| \\
&\geq |a_0|\{\left|t + \frac{\rho z}{a_0}\right| - (\frac{\rho + a_0}{|a_0|})|z|\} \\
&> 0
\end{aligned}$$

if

$$(\frac{\rho + a_0}{|a_0|})|z| < \left|t + \frac{\rho z}{a_0}\right|.$$

It is easy to see that the region defined by $(\frac{\rho + a_0}{|a_0|})|z| < \left|t + \frac{\rho z}{a_0}\right|$ is precisely the disk

$$\{z; \left|z - (\frac{\rho}{2\rho + a_0})t\right| < (\frac{\rho + a_0}{2\rho + a_0})t \ \}.$$

Hence it follows that F(z) and therefore f(z) does not vanish in the disk

$$\left|z - (\frac{\rho}{2\rho + a_0})t\right| < (\frac{\rho + a_0}{2\rho + a_0})t.$$

That proves Theorem 1.

Proof of Theorem 2: Since the function $f(z)=\sum_{j=0}^{\infty} a_j z^j$ is analytic for $|z|\le t$, it follows that the function

$$g(z)=f(tz)=\sum_{j=0}^{\infty} a_j t^j z^j$$

is analytic for $|z|\le 1$. Consider the function

$$\begin{aligned}
G(z) &= (z-1)g(z) \\
&= (z-1)(a_0+a_1tz+a_2t^2z^2+\ldots\ldots) \\
&= -a_0+(a_0-ta_1)z+\sum_{j=2}^{\infty}(a_{j-1}t^{j-1}-a_jt^j)z^j \\
&= -a_0+(a_0-ta_1)z+zF(z),
\end{aligned}$$

where

$$F(z)=\sum_{j=2}^{\infty}(a_{j-1}t^{j-1}-a_jt^j)z^{j-1}.$$

Now by hypothesis

$$\left|(\alpha_1-t\alpha_2)+(\alpha_2-t\alpha_3)z+(\alpha_3-t\alpha_4)z^2+\ldots..\right|\le\frac{M_1}{t}\text{ for } |z|=t$$

and

$$\left|(\beta_1-t\beta_2)+(\beta_2-t\beta_3)z+(\beta_3-t\beta_4)z^2+\ldots..\right|\le\frac{M_2}{t}\text{ for } |z|=t.$$

or

$$t\left|(\alpha_1-t\alpha_2)+(\alpha_2-t\alpha_3)z+(\alpha_3-t\alpha_4)z^2+\ldots..\right|\le M_1\text{ for } |z|=t$$

and

$$t\left|(\beta_1-t\beta_2)+(\beta_2-t\beta_3)z+(\beta_3-t\beta_4)z^2+\ldots..\right|\le M_2\text{ for } |z|=t.$$

Equivalently

$$\left|(\alpha_1-t\alpha_2)z+(\alpha_2-t\alpha_3)z^2+(\alpha_3-t\alpha_4)z^3+\ldots..\right|\le M_1\text{ for } |z|=t$$

and

$$\left|(\beta_1-t\beta_2)z+(\beta_2-t\beta_3)z^2+(\beta_3-t\beta_4)z^3+\ldots..\right|\le M_2\text{ for } |z|=t.$$

Replacing z by tz in the above inequalities , we get

$$\left|(\alpha_1-t\alpha_2)tz+(\alpha_2-t\alpha_3)t^2z^2+(\alpha_3-t\alpha_4)t^3z^3+\ldots..\right|\le M_1\text{ for } |z|=1$$

and

$$\left|(\beta_1-t\beta_2)tz+(\beta_2-t\beta_3)t^2z^2+(\beta_3-t\beta_4)t^3z^3+\ldots..\right|\le M_2\text{ for } |z|=1.$$

Hence , for $|z|=1$,

$$\begin{aligned}
|F(z)| &= \left|\sum_{j=2}^{\infty}(a_{j-1}t^{j-1}-a_jt^j)z^{j-1}\right| \\
&= \left|\sum_{j=2}^{\infty}\left[\{\alpha_{j-1}t^{j-1}-\alpha_jt^j\}e^{i\phi}z^{j-1}+\{\beta_{j-1}t^{j-1}-\beta_jt^j\}e^{i\psi}z^{j-1}\right]\right| \\
&\le \left|(\alpha_1-t\alpha_2)tz+(\alpha_2-t\alpha_3)t^2z^2+(\alpha_3-t\alpha_4)t^3z^3+\ldots..\right| \\
&\quad +\left|(\beta_1-t\beta_2)tz+(\beta_2-t\beta_3)t^2z^2+(\beta_3-t\beta_4)t^3z^3+\ldots..\right| \\
&\le M_1+M_2.
\end{aligned}$$

Clearly F(z) is analytic for $|z| \le 1$ and F(0)=0. Therefore applying Schwarz lemma to the function F(z), we get

$$|F(z)| \le (M_1 + M_2)|z| \text{ for } |z| \le 1.$$

Hence for $|z| \le 1$,

$$|G(z)| \ge |-a_0 + (a_0 - ta_1)z| - (M_1 + M_2)|z|^2$$

$$\ge |a_0| - |a_0 - ta_1||z| - (M_1 + M_2)|z|^2$$

$$= (M_1 + M_2)(A - |z|)(|z| + B)$$

where

$$A = \frac{1}{2(M_1 + M_2)}\left\{-|a_0 - ta_1| + \sqrt{|a_0 - ta_1|^2 + 4|a_0|(M_1 + M_2)}\right\}$$

and

$$B = \frac{1}{2(M_1 + M_2)}\left\{|a_0 - ta_1| + \sqrt{|a_0 - ta_1|^2 + 4|a_0|(M_1 + M_2)}\right\}.$$

Clearly $\delta > 0$ and for $|z| \le 1$, $|G(z)| > 0$ if $|z| < \gamma$. Hence it follows that G(z) and therefore f(z) does not vanish in $|z| < At$, which is equivalent to the desired result. This completes the proof of Theorem 2.

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